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Some finiteness results concerning separation in graphs

R. Halin

Mathematisches Seminar der Universität Hamburg, Bundesstraße 55, W-2000 Hamburg 13, Germany

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1. Introduction

Let G be a graph and J a subgraph of G or a subset of $V(G)$. Then ∂J denotes the set of all vertices of $G-J$ which are adjacent to at least one vertex of J. A subset C of $V(G)$ is called a *cut* of G if there are at least two connected components H, J of $G-C$ with $\partial H = \partial J = C$; connected components with the latter property will be referred to as C-*components* of G. C is an a, b-*cut* (for any pair of vertices a, b of G) if a, b belong to different C-components. Thus $C \subseteq V(G)$ is an a, b-cut if and only if C separates a, b and is minimal (with respect to inclusion) with regard to this property. The sets of all cuts of G will be denoted by $\mathscr{C}_G$, and $\mathscr{C}_G(a, b)$ shall denote the set of all a, b-cuts in G. If n is a cardinal, $\mathscr{C}_G(a, b; n)$ and $\mathscr{C}_G(a, b; \leq n)$ will denote the set of all a, b-cuts with n elements (at most n elements, respectively); the terms $\mathscr{C}_G(a, b; <n)$, $\mathscr{C}_G(a, b; \geq n)$, $\mathscr{C}_G(a, b; >n)$ will be used analogously. ω denotes the smallest infinite ordinal and, at the same time, the cardinality of $\mathbb{N}$ ($\mathbb{N}$ is the set of positive integers).

The main and basic result of the present note is the following: If $\mathscr{C}_G(a, b; <\omega)$ contains an infinite subset $\mathscr{M}$, then a $U \in \mathscr{C}_G(a, b; \omega)$ and a sequence $C_1, C_2, C_3, \ldots$ of elements of $\mathscr{M}$ can be found so that

$$(U \cap C_1) \subseteq (U \cap C_2) \subseteq (U \cap C_3) \subseteq \cdots$$

is an infinite ascending sequence whose union is U.

In particular we see that, if $\mathscr{C}_G(a, b; \geq\omega)$ is empty, then there must be a finite n such that $\mathscr{C}_G(a, b) = \mathscr{C}_G(a, b; \leq n)$; further it is clear from our theorem that always $\mathscr{C}_G(a, b; \leq n)$ (for $n \in \mathbb{N}$) must be finite (since the cardinalities of the C_n cannot be bounded by an integer).

This statement in turn implies several further finiteness results on cuts of graphs with specific properties and bounded finite cardinalities. For instance let

$\mathscr{C}_G(\infty, \infty)$ denote the set of finite cuts C of G such that there are at least two infinite C-components, and let κ_∞ be the minimum of the cardinalities of the elements in $\mathscr{C}_G(\infty, \infty)$ (if there are any). Jung and Watkins [6, Theorem 3.1] showed that any vertex v of finite degree in G is contained in only finitely many cuts $\in \mathscr{C}_G(\infty, \infty)$ of order κ_G. We shall easily obtain the stronger result that v is in only finitely many cuts $\in \mathscr{C}_G(\infty, \infty)$ with order $\leq n$, for any given $n \in \mathbb{N}$.

It is not difficult to carry over these finiteness results to edge-cuts and mixed cuts (i.e. cuts which contain vertices and edges). In this way results of Stallings [7] and Dunwoody [1] can be sharpened with simpler proofs. (In the context of these papers the study of edge-cuts has a more auxiliary character, but the results are of interest also in their own right.)

In the last section it is shown that in rayless graphs (i.e. graphs without infinite paths) always the sets $\mathscr{C}_G(a, b; <\omega)$ are finite. In such graphs, on the other hand, $\mathscr{C}_G(a, b; \omega)$ may be uncountable. In the proof a representation derived from simplicial decompositions is used. (See the monograph of Diestel [1] or Chap. 10 of Halin [3] for the theory of simplicial decompositions of graphs.)

2. A theorem on sequences of finite cuts

$T \subseteq V(G)$ *separates* the graph G if $G - T$ has at least two connected components. If a is in $V(G)$, we denote the connected component of G containing a by $K_G(a)$. T separates the vertices a and b (in G) if $K_{G-T}(a) \neq K_{G-T}(b)$.

Lemma 1. *If T separates a, b in G, then there is an a, b-cut C of G contained in T.*

Proof. Also $T' := \partial K_{G-T}(a)$ separates a, b. Then $\partial K_{G-T'}(b)$ is the desired a, b-cut. □

Not every $T \subseteq V(G)$ which separates a graph G contains a minimal subset (with respect to inclusion) with this property. For instance choose, for every infinite subset S of $\mathbb{N}$, a vertex v_S which is joined (by edges) to the members of S. The arising bipartite graph does not contain a minimal separating set of vertices at all, though we know by Lemma 1 that each pair of non-adjacent vertices a, b admits an a, b-cut.

$T \subseteq V(G)$ is called a *partial a, b-cut* of G if there are disjoint connected subgraphs H_a, H_b with $a \in V(H_a)$, $b \in V(H_b)$ such that $\partial H_a \supseteq T$, $\partial H_b \supseteq T$ and there is no edge between H_a and H_b. In $G - T$ the vertices not in $H_a \cup H_b$ separate a, b; by Lemma 1 we find an a, b-cut C' in $G - T$ disjoint from $H_a \cup H_b$, and then $T \cup C'$ is an a, b-cut in G. Therefore we have, as a certain counterpart (or dual) of Lemma 1:

Every partial a, b-cut of G can be extended to an a, b-cut of G. From this we see that a vertex x of G is contained in a cut of G if and only if the neighbours of x do not induce a complete subgraph of G.

Now the main result of this paper is proved.

Theorem 1. *Let $a \neq b$ be vertices of a graph G, and let $\mathcal{M}$ be an infinite set of finite a, b-cuts in G. Then there exists a countable a, b-cut U in G and an infinite sequence $C_1, C_2, C_3, \ldots$ of elements of $\mathcal{M}$ such that, with $U \cap C_i =: U_i$, we have*

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$$

and

$$U = \bigcup_{n=1}^{\infty} U_n.$$

Proof. Without loss of generality we may assume $\mathcal{M}$ to be countable. Let M denote the union of the elements of $\mathcal{M}$; of course M is countable too. So we can choose an enumeration

$$x_1, x_2, x_3, \ldots$$

of the vertices in M.

Consider an a, b-path W_1 in G. As W_1 meets every $C \in \mathcal{M}$, there is an infinite subset $\mathcal{M}_1$ of $\mathcal{M}$ such that all the cuts of $\mathcal{M}_1$ contain the same vertex u_1 of W_1; let $W_1, \mathcal{M}_1, u_1$ be chosen in such a way that $u_1 = x_{i_1}$ with smallest possible i_1.

Now assume that, for an integer $n \geqslant 1$, distinct vertices $u_1, \ldots, u_n$ and an infinite subset $\mathcal{M}_n$ of $\mathcal{M}$ have been determined in such a way that $u_1, \ldots, u_n$ are contained in each cut $C \in \mathcal{M}_n$. Of course $\{u_1, \ldots, u_n\}$ is not an a, b-cut; otherwise every $C \in \mathcal{M}_n$ would coincide with this set. Hence there is an a, b-path W_{n+1} which avoids $u_1, \ldots, u_n$. Since W_{n+1} meets all the cuts in $\mathcal{M}_n$, there is an infinite subset $\mathcal{M}_{n+1}$ of $\mathcal{M}_n$ such that the elements of $\mathcal{M}_{n+1}$ share a vertex u_{n+1} of W_{n+1}; let $W_{n+1}, \mathcal{M}_{n+1}, u_{n+1}$ be chosen in such a way that u_{n+1} appears in the given enumeration of M as an $x_{i_{n+1}}$ with smallest possible i_{n+1}.

In this way we get an infinite sequence of distinct vertices $u_1, u_2, u_3, \ldots$ and infinite subsets $\mathcal{M}_n$ of $\mathcal{M}$ such that $\mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \cdots$ and each $C \in \mathcal{M}_n$ contains $u_1, \ldots, u_n$; further each u_n is chosen smallest possible in the given enumeration of M. Put $U' = \{u_1, u_2, u_3, \ldots\}$. We claim that U' separates a, b in G.

Otherwise there is an a, b-path $W \subseteq G$ with $V(W) \cup U' = \emptyset$. Put

$$h = \max(i \in \mathbb{N} \mid x_i \in M \cap V(W)).$$

There is a u_k in U' with $u_k = x_{i_k}$ and $i_k > h$. Now W is an a, b-path avoiding $u_1, \ldots, u_{k-1}$, and all the vertices of M in W precede u_k in the given enumeration; one of these vertices must be contained in infinitely many $C \in \mathcal{M}_{k-1}$. But then u_k had been selected in the wrong way.

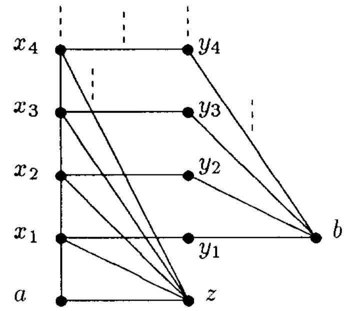

Fig. 1.

By Lemma 1 we find an a, b-cut $U \subseteq U'$. U cannot be finite because no subset $\{u_1, \ldots, u_n\}$ of U' separates a, b. For any $C \in \mathcal{M}$ let $m(C)$ denote the smallest n such that $u_n \in U - C$. Then we determine the sequence $C_1, C_2, C_3, \ldots$ by choosing $C_1 \in \mathcal{M}_1$ arbitrarily and each $C_{n+1} \in \mathcal{M}_{m(C_n)}$. U and the C_i satisfy the requirements of the theorem. □

Remark. Examples show that in the last proof it may be really necessary to take a proper subset of U' as U. In Fig. 1 vertex z must be contained in every finite a, b-cut. If $C_n = \{z, y_1, \ldots, y_n, x_{n+1}\}$ $(n = 1, 2, \ldots.)$, then the construction of our proof yields $U' = \{z, y_1, y_2, \ldots.\}$ (based on any enumeration of $\bigcup C_n$), but U must be chosen as the set $\{y_1, y_2, y_3, \ldots\}$.

3. Finiteness results for cuts with certain restrictions

Clearly the C_n constructed in Theorem 1 cannot be of bounded size. So, if we let $\mathcal{M} = \mathscr{C}_G(a, b; \leq n)$, we find the following.

Corollary 1. *For any $n \in \mathbb{N}$, any graph G and non-adjacent vertices a, b of G we have*:

$$\mathscr{C}_G(a, b; \leq n) \text{ is finite.}$$

Corollary 1 is the base for what follows in this and the next section. By the way, if we restrict the proof of Theorem 1 to the assertion of Corollary 1, the line of argument becomes even simpler.

By Corollary 1 we also see that always $\mathscr{C}_G(a, b; <\omega)$ is at most countable. That it may be infinite is shown by the graph of Fig. 2. It also shows that $\mathscr{C}_G(a, b; \omega)$ may be of cardinality 2^ω (consider the cuts with vertices c_i, d_j).

If a vertex x of G is in a cut C and H, J are C-components of G, then there are neighbours a, b of x in H and J, respectively. We see that any cut C of G which contains x is in $\mathscr{C}_G(a, b)$ for some pair of neighbours a, b of x. Therefore from Corollary 1 we immediately have the following two results.

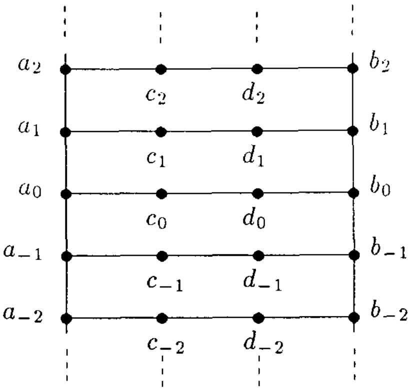

Fig. 2.

Corollary 2. *If x is of finite degree in G and n is any natural number, then x belongs to only finitely many cuts of order $\leqslant n$.*

Corollary 3. *If x is of degree $\leqslant \delta$ where δ is an arbitrary infinite cardinal, then x is in at most δ finite cuts.*

By Corollary 2 it is clear that every vertex x of finite degree belongs only to finitely many members of $\mathscr{C}_G(\infty, \infty)$ with order $\leqslant n$, n any given natural number. On the other hand in the locally finite example of Fig. 2 every vertex x is contained in infinitely (more exactly: countably) many elements of $\mathscr{C}_G(\infty, \infty)$ (which of course cannot have bounded size).

Next we show the following 'Descending Chain Condition'.

Corollary 4. *Let $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ be a descending sequence of non-empty connected induced subgraphs of a graph G such that there is an $h \in \mathbb{N}$ with*

$$|\partial H_n| \leqslant h$$

for all n. Then either $\bigcap_{n\in\mathbb{N}} H_n = \emptyset$ or there is a $k \in \mathbb{N}$ such that $H_n = H_k$ for all $n \geqslant k$.

Proof. Assume that there is a vertex x in $\bigcap_{n\in\mathbb{N}} H_n$. Without loss of generality we may assume that $V(H_1) \cup \partial H_1$ is properly contained in $V(G)$; let y be a vertex of G not in $H_1 \cup \partial H_1$. Of course each ∂H_i separates x and y. By drawing additional edges between y and ∂H_1 and between ∂H_i and ∂H_{i+1} for $i = 1, 2, \ldots$ (if necessary) we achieve that all the ∂H_i are x, y-cuts in the extended graph. (The additional edges may also extend the H_i properly, but this does not affect their intersections.) From Corollary 1 we find that the ∂H_n coincide for all $n \geqslant k$ for k sufficiently large. But then also the H_n for $n \geqslant k$ coincide. $\square$

The example of Fig. 2 again shows that it does not suffice in Corollary 4 to assume only that all ∂H_n are finite.

Let C, C' be cuts of G. We say that C crosses C' if there is a pair of vertices in C' which are separated by C in G.

Lemma 2. *If C crosses C', then C' crosses C.*

Proof. Assume that C' does not cross C. Then there exists a component H' of $G-C'$ such that $C \subseteq V(H') \cup C'$. There is a C'-component H'' with $\partial H'' = C'$, $H'' \neq H'$. Then any two vertices of C' not in C can be joined by a path through H'' which does not meet C; we get a contradiction to the assumption that C crosses C'. □

Therefore we may say that C, C' *cross* (each other) in G.

Lemma 3. *If C crosses C' then there exist a, b in C' such that C is an a, b-cut of G.*

Proof. Let H, J be two distinct C-components of G. If $V(H) \cap C'$ or $V(J) \cap C'$ were empty, then any pair of vertices $\in C - C'$ could be connected by a path through H or J, respectively, which does not meet C', contradicting Lemma 2. Therefore we find vertices $a \in V(H) \cap C'$, $b \in V(J) \cap C'$ and by choice of H, J we see that C is an a, b-cut. □

As an immediate consequence of our last two lemmas together with Corollary 1 we have the following.

Corollary 5. *If C is a finite cut of G and n any positive integer, then there are only finitely many cuts of order $\leqslant n$ which cross C.*

In general we find that each finite cut is crossed by at most countably many finite cuts.

In Fig. 2 each finite cut is crossed by infinitely many different other finite cuts and by uncountably many countable cuts. Especially there are infinitely many finite a_0, b_0-cuts which cross each other. (Such a system is, for instance, formed by the cuts $C_k = \{a_{-1}, c_0, \ldots, c_k, d_{k+1}, a_{k+2}\}$.) This shows that in Theorem 1 the C_n in general cannot be constructed in such a way that they form a monotonical sequence in the lattice of a, b-cuts (as defined in [3, Chap. 11, §1]).

4. Edge-cuts and mixed cuts

Let $G=(V, E)$ be a graph, a, b distinct vertices of G. A subset C of $V \cup E$ *separates* a, b in G if $K_{G-C}(a) \neq K_{G-C}(b)$; here $G - C$ is the graph $(V - C,$

$E - C$). $C \subseteq V \cup E$ is a *mixed a, b-cut* if C separates a, b in G, but no proper subset of C also separates a, b. By $\mathscr{C}_G^m(a, b)$ we denote the set of all mixed a, b-cuts of G. A *mixed cut* of G is an element of $\mathscr{C}_G^m(a, b)$ for any pair of distinct vertices a, b of G; the set of mixed cuts of G is denoted by $\mathscr{C}_G^m$. An element C of $\mathscr{C}_G^m$ with $C \cap V = \emptyset$ is called an *edge-cut* of G. Let $\mathscr{C}_G^e$, $\mathscr{C}_G^e(a, b)$ denote the set of edge-cuts of G and the set of a, b-edge-cuts in G, respectively. In contrast to the cuts (as considered in Sections 1–3), the edge-cuts are the minimal (with respect to inclusion) separating edge sets of a graph, and the deletion of any edge-cut of G leaves exactly two components, which we call the *leaves* of G with respect to C (briefly, the *C-leaves*).

If H is a subgraph of G let us denote by δH the set of all edges of G joining a vertex of H with a vertex outside H. If H is a leaf of G (with respect to some C) then $\delta H = C$; so a leaf of G determines the cut to which it belongs uniquely. The leaf H also determines its 'counterpart' $\bar{H} := G - H$; $H, \bar{H}$ are the two C-leaves for $C := \delta H = \delta \bar{H}$.

Now let G^* be the graph arising from G by inserting a vertex v_e on each edge e of G. (G^* is also known as the subdivision-graph of G.) Define the mapping

$$\varphi : V \cup E \rightarrow V(G^*)$$

by letting $\varphi(x) = x$ for $x \in V$, $\varphi(e) = v_e$ for $e \in E$. Then φ obviously is a bijection. For any distinct $a, b \in V$ we easily find the following.

Lemma 4. *φ induces a bijection of $\mathscr{C}_G^m(a, b)$ onto $\mathscr{C}_{G^*}(a, b)$.*

A cut of G^* is called *trivial* if it consists of the two end-vertices of an edge $e \in E$ and is not a cut of G. It is easy to see that every nontrivial cut of G^* is in some $\mathscr{C}_{G^*}(a, b)$ with $a, b \in V$. Therefore, by Lemma 4, φ induces a bijection of $\mathscr{C}_G^m$ onto the set of nontrivial cuts of G^*. Under φ the edge-cuts of G correspond to certain cuts of G^*, namely those which contain only vertices of the form v_e, $e \in E$. From these considerations Corollaries 1–5 immediately imply the following statements.

Corollary 6. *For any $n \in \mathbb{N}$, any graph G and vertices a, b of G there are only finitely many a, b-edge-cuts in G of cardinality $\leqslant n$.*

Corollary 7. *Any edge e of G belongs to only finitely many edge-cuts of cardinality $\leqslant n$, where n is any given natural number.*

(Mind that v_e has degree 2 in G^*.)

Stallings [7] and Dunwoody [2] consider edge-cuts C of a graph G such that both leaves of C are infinite and C is of minimal finite cardinality; then C and also its leaves are called *narrow*. Statement 2.5 of [2] says that any edge of a graph G can occur only in finitely many narrow edge-cuts of G. This result is generalize results of [7] and [2] on narrow edge-cuts and leaves (see [2, 2.1 and 2.6]); they are immediate consequences of Corollaries 5 and 6 in connection with Lemma 4.

Corollary 8. *Let $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ be a descending sequence of leaves of a graph G such that there is an $h \in \mathbb{N}$ with*

$$|\delta H_n| \leq h$$

for all n. Then either $\bigcap_{n\in\mathbb{N}} H_n = \emptyset$ or there is a $k \in \mathbb{N}$ such that $H_n = H_k$ for all $n \geq k$.

Corollary 9. *If C is a finite edge-cut of G and n any integer, then there are only finitely many edge-cuts of order $\leq n$ which cross C.*

Here we define two edge-cuts C, C' to cross (each other) if the cuts corresponding to C, C' under φ cross. It is easy to see that the following statements (for edge-cuts C, C') are equivalent:

(a) C, C' cross;
(b) both leaves of C meet both leaves of C';
(c) each leaf H of C is separated by the edges of C' in H;
(d) there is no inclusion between any leaf of C and any leaf of C';
(e) any leaf of C and any leaf of C' have non-empty intersection.

5. Cuts in rayless graphs

In this section we shall show that if the situation of Theorem 1 is given then there must be in G a ray (i.e. a one-way infinite path). The proof is based on the following representation of rayless graphs given in [2] and [3].

A connected graph G is *rayless* if and only if there is a well-ordered family of finite subgraphs $(G_\lambda)_{\lambda<\sigma}$ (here σ is an ordinal >0) such that the following conditions are fulfilled:

(1) $G = \bigcup_{\lambda<\sigma} G_\lambda$;

(2) For each τ, $0<\tau<\sigma$, there is a smallest $\tau_- < \tau$ such that $T_\tau := (\bigcup_{\lambda<\tau} G_\lambda) \cap G_\tau$ is properly contained in both G_{τ_-} and G_τ;

(3) Each T_τ is a 'pseudo-simplex' in G, which means that any two vertices $x \neq y$ of T are adjacent or have Menger number (or local connectivity number) $\mu_G(x, y) \geq \omega$;

(4) Each G_λ is 'pseudo-prime' in G (i.e. it is not separated by a pseudo-simplex in G);

(5) The decomposition tree of this representation (i.e. the tree with the vertices $\lambda < \sigma$ and the edges $[\tau, \tau_-]$, $0 < \tau < \sigma$) is rayless.

Condition (4) makes the given representation 'canonical', which means that the members G_λ (though not their order) are uniquely determined. For the characterization of rayless graphs also conditions (1)–(3) and (5) would be sufficient. (Therefore, if we do not insist on Condition (4), appropriate finite sections of the given representation may be joined to a new member, and we again get a decomposition of the kind described above.)

Now we show the following.

Theorem 2. *If G is a rayless graph and a, b are distinct non-adjacent vertices of G, then $\mathscr{C}_G(a, b; <\omega)$ is finite.*

Proof. Of course it is no restriction to assume G to be connected. Let a decomposition of G as described above be given. Let λ, $\tau < \sigma$ be chosen such that a is in G_λ, b in G_τ, and let F be the union of all G_ν where ν is on the λ, τ-path in the decomposition tree of the given representation. Of course F is a finite induced subgraph of G, and we may write

$$G = F \cup \bigcup_{i \in I} H_i$$

where the H_i correspond to the connected components of $G - F$. More precisely: If H_i' $(i \in I)$ are the connected components of $G - F$, then (for each $i \in I$) H_i is the subgraph of G induced by H_i' and $\partial H_i'$. Then for each $i \in I$, by the properties of the given decomposition, $T_i := H_i \cap F$ is a pseudo-simplex in G.

It suffices to show that every finite a, b-cut lies in F. Assume, on the contrary, that the finite a, b-cut C contains a vertex x in $H_i - T_i$ for an $i \in I$. There exists an a, b-path P in G with $V(P) \cap C = \{x\}$; without loss of generality we may assume that P is an induced subgraph of G. It has a last vertex y before x and a first vertex z after x in common with T_i (possibly $a = y$ or $b = z$). Clearly $y \neq z$, and y, z are non-adjacent by choice of P. But then $\mu_G(y, z) \geq \omega$, and we find a y, z-path Q in G which avoids C. By walking on P from a to y, then on Q from y to z, and finally on P from z to b, we find a path from a to b which does not meet C. By this contradiction our proof is complete. $\square$

In this proof the raylessness of G (more precisely: Condition 5)) was not really used. So we see that the assertion of Theorem 2 is true for all graphs admitting a decomposition of the kind considered above with finite members G_λ and fulfilling only conditions (1)–(3).

Theorem 2 does not extend to infinite a, b-cuts: The union of δ internally disjoint a, b-paths of length 3 (where δ is any infinite cardinal) is a rayless graph with 2^δ distinct a, b-cuts of order δ.

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